Speculations on the integral formula $\int \frac{x^n dx}{\sqrt{aa-2bx-cxx}}$ where at the same time extraordinary observations on continued fractions occur *

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§1 Let us start from the simplest case in which $n=0$ and find the integral of the formula

$$\frac{dx}{\sqrt{aa-2bx+cxx}},$$

which, having put $x=\frac{b+z}{c}$, goes over into this one

$$\frac{dz}{\sqrt{aacc-bbc+czz}},$$

where two cases must be distinguished, depending on whether c was a positive or negative quantity.
Therefore, first let $c=+ff$ and our formula will become

$$\frac{dz}{f\sqrt{aaff-bb+zz}},$$

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whose integral is

$$\frac{1}{f}\log\frac{z+\sqrt{aaff-bb+zz}}{C},$$

and hence our integral will be

$$\frac{1}{\sqrt{c}}\log\frac{cx-b+\sqrt{aac-2bcx+ccxx}}{C},$$

which, taken in such a way that it vanishes for $x=0$, will therefore be

$$\frac{1}{\sqrt{c}}\log\frac{cx-b+\sqrt{c(aa-2bx+cxx)}}{-b+a\sqrt{c}}.$$

But if c was a negative quantity, say $c=-gg$, the differential formula expressed in terms of z will be

$$\frac{dz}{g\sqrt{aagg+bb-zz}},$$

whose integral is

$$\frac{1}{g}\arcsin\frac{z}{\sqrt{aagg+bb}}+C,$$

whence the integral, taken in such a way that it vanishes for $x=0$, will become

$$=\frac{1}{g}\arcsin\frac{cx-b}{\sqrt{aagg+bb}}+\frac{1}{g}\arcsin\frac{b}{\sqrt{aagg+bb}}.$$

§2 Now let Π denote the value of the integral formula $\int\frac{dx}{\sqrt{aa-2bx+cxx}}$ taken in such a way that it vanishes for $x=0$, no matter whether c was a positive or negative quantity; and if $c=+ff$, it will be, as we saw,

$$\Pi=\frac{1}{f}\log\frac{ffx-b+f\sqrt{aa-2bx+ffxx}}{af-b};$$

in the other case in which $c=-gg$, it will be

$$\Pi=-\frac{1}{g}\arcsin\frac{ggx+b}{\sqrt{aagg+bb}}+\frac{1}{g}\arcsin\frac{b}{\sqrt{aagg+bb}}$$

or, having contracted both arcs, we will have

$$\Pi = \frac{1}{g} \arcsin \frac{bg\sqrt{(aa-2bx-ggxx)} - abg - ag^3x}{aagg+bb}.$$

Therefore, since we will soon show that the integration of the general integral formula $\int \frac{x^n dx}{\sqrt{aa-2bx+cxx}}$ can always be reduced to the case $n=0$, if just n was a positive integer number, all these integral can be expressed in terms of that value Π.

§3 Now let us attribute a constant value of such a kind to the variable quantity x after the integration that the irrational formula $\sqrt{aa-2bx+cxx}$ becomes zero, which happens, if one takes

$$x = \frac{b \pm \sqrt{bb-aac}}{c},$$

and hence in two cases. For each of both cases let us put that the function Π goes over into Δ such that in the case $c = ff$ is

$$\Delta = \frac{1}{f} \log \frac{\sqrt{bb-aaff}}{af-b} = \frac{1}{f} \log \sqrt{\frac{b+af}{b-af}},$$

but for the other case, in which $c = -gg$,

$$\Delta = \frac{1}{g} \arcsin \frac{\pm ag\sqrt{bb+aagg}}{aagg+bb} = \frac{1}{g} \arcsin \frac{ag}{\sqrt{bb+aagg}}.$$

Indeed, in the following we will mainly contemplate these values Δ, i.e. the cases in which the formula $\sqrt{aa-2bx+cxx}$ vanishes.

§4 Now, proceeding to the following case, let us consider the formula

$$s = \sqrt{aa-2bx+cxx} - a,$$

that it vanishes for $x=0$, and since

$$ds = \frac{-bdx + cxdx}{\sqrt{aa-2bx+cxx}},$$

vice versa by integrating it will be

$$c\int\frac{xdx}{\sqrt{aa-2bx+cxx}}=b\int\frac{dx}{\sqrt{aa-2bx+cxx}}+s,$$

whence we conclude

$$\int\frac{xdx}{\sqrt{aa-2bx+cxx}}=\frac{b}{c}\Pi+\frac{\sqrt{aa-2bx+cxx}-a}{c};$$

hence if after the integration we set $x=\frac{b\pm\sqrt{bb-aac}}{c}$, in which case $\sqrt{aa-2bx+xcc}=0$ and $\Pi=\Delta$, it will be

$$\int\frac{xdx}{\sqrt{aa-2bx+cxx}}=\frac{b}{c}\Delta-\frac{a}{c}.$$

§5 Further, let us take

$$s=x\sqrt{aa-2bx+cxx};$$

it will be

$$ds=\frac{aadx-3bxdx+2cxxdx}{\sqrt{aa-2bx+cxx}},$$

whence vice versa by integration one concludes

$$2c\int\frac{xxdx}{\sqrt{aa-2bx+cxx}}=3b\int\frac{xdx}{\sqrt{aa-2bx+cxx}}-aa\int\frac{dx}{\sqrt{aa-2bx+cxx}}+s,$$

whence for the case $\sqrt{aa-2bx+cxx}=0$ we immediately deduce

$$\int\frac{xxdx}{\sqrt{aa-2bx+cxx}}=\frac{3bb-aac}{2cc}\Delta-\frac{3ab}{2cc}.$$

§6 Now, going to ascend to higher powers, let us set

$$s=xx\sqrt{aa-2bx+cxx},$$

and since hence

$$ds=\frac{2aaxdx-5bxxdx+3cx^3dx}{\sqrt{aa-2bx+cxx}},$$

it will be

$$3c\int\frac{x^3dx}{\sqrt{aa-2bx+cxx}}=5b\int\frac{xxdx}{\sqrt{aa-2bx+cxx}}-2aa\int\frac{xdx}{\sqrt{aa-2bx+cxx}}+s$$

and hence further for the case, in which one sets $x=\frac{b\pm\sqrt{bb-aac}}{c}$ after the integration, one will have

$$\begin{aligned}\int\frac{x^3dx}{\sqrt{aa-2bx+cxx}} &= \frac{5b^3-3aabc}{2c^3}\Delta-\frac{15abb}{6c^3}+\frac{2a^3}{3cc}\\ &=\left(\frac{5b^3}{2c^3}-\frac{3aab}{2cc}\right)\Delta-\frac{5abb}{2c^3}+\frac{2a^3}{3cc}.\end{aligned}$$

In like manner, let

$$s=x^3\sqrt{aa-2bx+cxx},$$

and since hence

$$ds=\frac{3aaxxdx-7bx^3dx+4cx^4dx}{\sqrt{aa-2bx+cxx}},$$

vice versa by integration it will be

$$4c\int\frac{x^4dx}{\sqrt{aa-2bx+cxx}}=7b\int\frac{x^3dx}{\sqrt{aa-2bx+cxx}}-3aa\int\frac{xxdx}{\sqrt{aa-2bx+cxx}}+s;$$

therefore, then for the case in which $\sqrt{aa-2bx+cxx}=0$ we will have

$$\int\frac{x^4dx}{\sqrt{aa-2bx+cxx}}=\left(\frac{35b^4}{8c^4}-\frac{15aabb}{4c^3}+\frac{3a^4}{8cc}\right)\Delta-\frac{35ab^3}{8c^4}+\frac{55a^3b}{24c^3}.$$

§8 But that the structure in these formulas can be explored more conveniently, let us exhibit each one in terms of products, as they result in order, without any abbreviation and represent the found integral formulas this way:

$$\int \frac{dx}{\sqrt{aa-2bx+cxx}} = \Delta,$$

$$\int \frac{xdx}{\sqrt{aa-2bx+cxx}} = \frac{b}{c}\Delta - \frac{a}{c},$$

$$\int \frac{xxdx}{\sqrt{aa-2bx+cxx}} = \left(\frac{1\cdot 3bb}{1\cdot 2cc} - \frac{aa}{1\cdot 2c}\right)\Delta - \frac{1\cdot 3ab}{1\cdot 2cc},$$

$$\int \frac{x^3dx}{\sqrt{aa-2bx+cxx}} = \left(\frac{1\cdot 3\cdot 5b^3}{1\cdot 2\cdot 3c^3} - \frac{1\cdot 3\cdot 3aab}{1\cdot 2\cdot 3cc}\right)\Delta - \frac{1\cdot 3\cdot 5abb}{1\cdot 2\cdot 3c^3} + \frac{1\cdot 2\cdot 2a^3}{1\cdot 2\cdot 3cc},$$

$$\int \frac{x^4dx}{\sqrt{aa-2bx+cxx}} = \left(\frac{1\cdot 3\cdot 5\cdot 7b^4}{1\cdot 2\cdot 3\cdot 4c^4} - \frac{1\cdot 3\cdot 5\cdot 6aabb}{1\cdot 2\cdot 3\cdot 4c^3} + \frac{1\cdot 3\cdot 3a^4}{1\cdot 2\cdot 3\cdot 4cc}\right)\Delta$$
$$- \frac{1\cdot 3\cdot 5\cdot 7ab^3}{1\cdot 2\cdot 3\cdot 4c^4} + \frac{1\cdot 5\cdot 11a^3b}{1\cdot 2\cdot 3\cdot 4c^3}.$$

§9 Now let us do this expansion in general by taking

$$s = x^n\sqrt{aa-2bx+cxx},$$

and since hence

$$ds = \frac{naax^{n-1}dx - (2n+1)bx^ndx + (n+1)cx^{n+1}dx}{\sqrt{aa-2bx+cxx}},$$

from this vice versa by integration one calculates

$$(n+1)c\int \frac{x^{n+1}dx}{\sqrt{aa-2bx+cxx}} = (2n+1)b\int \frac{x^ndx}{\sqrt{aa-2bx+cxx}}$$
$$-naa\int \frac{x^{n-1}dx}{\sqrt{aa-2bx+cxx}} + x^n\sqrt{aa-2bx+cxx}.$$

Therefore, if we had already found before

$$\int \frac{x^{n-1}dx}{\sqrt{aa-2bx+cxx}} = M\Delta - \mathfrak{M} \quad \text{and} \quad \int \frac{x^ndx}{\sqrt{aa-2bx+cxx}} = N\Delta - \mathfrak{N},$$

such that these two integral formulas are known, the following will be determined from them in such a way that

$$\int \frac{x^{n+1}dx}{\sqrt{aa-2bx+cxx}} = \left(\frac{(2n+1)bN}{(n+1)c} - \frac{naaM}{(n+1)c}\right)\Delta - \frac{(2n+1)b\mathfrak{N}}{(n+1)c} + \frac{nna\mathfrak{M}}{(n+1)c}.$$

Therefore, this way these integrations can be continued arbitrarily far, while from two the following one is formed by means of this rule, such that all these integrals depend either on logarithms or circular arcs, depending on whether the coefficient c was positive or negative. But it is manifest that those values can only be assigned, if the exponent n was a positive integer number.

§10 From the integral form just found, if after the integration one sets $x = \frac{b\pm\sqrt{bb-aac}}{c}$, whence $s=0$, it will be

$$naa\int \frac{x^{n-1}dx}{\sqrt{aa-2bx+cxx}}$$
$$= (2n+1)b\int \frac{x^{n}dx}{\sqrt{aa-2bx+cxx}} - (n+1)c\int \frac{x^{n+1}dx}{\sqrt{aa-2bx+cxx}};$$

hence, if for the sake of brevity we put

$$\int \frac{x^{n-1}dx}{\sqrt{aa-2bx+cxx}} = P, \quad \int \frac{x^{n}dx}{\sqrt{aa-2bx+cxx}} = Q,$$
$$\int \frac{x^{n+1}dx}{\sqrt{aa-2bx+cxx}} = R, \quad \int \frac{x^{n+2}dx}{\sqrt{aa-2bx+cxx}} = S \quad \text{etc.},$$

these quantities P, Q, R, S depend on each other in such a way that

$$\begin{aligned} naaP &= (2n+1)bQ - (n+1)cR, \\ (n+1)aaQ &= (2n+3)bR - (n+2)cS, \\ (n+2)aaR &= (2n+5)bS - (n+3)cT, \\ (n+3)aaS &= (2n+7)bT - (n+4)cU, \\ (n+4)aaT &- (2n+9)bU - (n+5)cW \end{aligned}$$

etc.

From these relations the following determinations are deduced

$$\frac{P}{Q} = \frac{(2n+1)b}{naa} - \frac{(n+1)c}{nnaQ:R},$$

$$\frac{Q}{R} = \frac{(2n+3)b}{(n+1)aa} - \frac{(n+2)c}{(n+1)aaR:S},$$

$$\frac{R}{S} = \frac{(2n+5)b}{(n+2)aa} - \frac{(n+3)c}{(n+2)aaS:T},$$

$$\frac{S}{T} = \frac{(2n+7)b}{(n+3)aa} - \frac{(n+4)c}{(n+3)aaT:U}$$

etc.;

therefore, hence it is plain that these fractions $\frac{P}{Q}$, $\frac{Q}{R}$, $\frac{R}{S}$ etc. are determined in terms of the others sufficiently conveniently.

§11 If now in one of these expressions the values just exhibited are successively substituted, we will obtain a continued fraction for the fraction $\frac{P}{Q}$, which will be

$$naa\frac{P}{Q} = (2n+1)b - \cfrac{(n+1)^2aac}{(2n+3)b - \cfrac{(n+2)^2aac}{(2n+5)b - \cfrac{(n+3)^2aac}{(2n+7)b - \cfrac{(n+4)^2aac}{(2n+9)b - \text{etc.}}}}}$$

and so we got to a rather nice and structured continued fraction, whose value can therefore always be expressed in terms of logarithms (if $c > 0$) or in terms of circular arc (if $c < 0$).

§12 Now let us take $n = 1$ and it will be

$$P = \int \frac{dx}{\sqrt{aa - 2bx + cxx}} = \Delta$$

and

$$Q = \int \frac{xdx}{\sqrt{aa - 2bx + cxx}} = \frac{b}{c}\Delta - \frac{a}{c},$$

which case gives us the following continued fraction

$$\frac{aac\Delta}{b\Delta - a} = 3b - \cfrac{4aac}{5b - \cfrac{9aac}{7b - \cfrac{16aac}{9b - \cfrac{25aac}{11b - \text{etc.}}}}}$$

which, for its elegance, it worth one's complete attention. But here it will be helpful to have noted, if c was a negative number, that then all numerators in this fraction become positive.

§12a But this continued fraction seems to be truncated by its head; hence, if at the top the term $b - aac$ is added, it becomes even nicer and its value is simplified. For, if for the sake of brevity that fraction is denoted by the letter S such that $S = \frac{aac\Delta}{b\Delta - a}$, having added that term, its value will be $b - \frac{aac}{S} = \frac{a}{\Delta}$ and so we will have

$$\frac{a}{\Delta} = b - \cfrac{aac}{3b - \cfrac{4aac}{5b - \cfrac{9aac}{7b - \cfrac{16aac}{9b - \cfrac{25aac}{11b - \text{etc.}}}}}}$$

which expression is even more memorable since there is no other way yet to find the value of such a continued fraction a priori.

§13 Now let us expand the two cases mentioned above separately, and it will be convenient to distinguish them carefully. Therefore, first let $c = ff$ and above we found that it will be

$$\Delta = \frac{1}{f} \log \frac{\sqrt{bb - aaff}}{af - b},$$

where the square root sign is to be understood ambiguously. Therefore, most importantly it is necessary that $bb > aaff$, since otherwise this expression

would become imaginary; therefore, two cases occur, depending on whether b was a positive or negative quantity.
In the first case in which $b > 0$ and hence $b > af$, it is evident that the square root sign must have the sign $-$, that

$$\Delta = \frac{1}{f}\log\frac{\sqrt{bb-aaff}}{b-af} = \frac{1}{2f}\log\frac{b+af}{b-af},$$

and we will have this summation

$$\frac{2af}{\log\frac{b+af}{b-af}} = b - \cfrac{aaff}{3b - \cfrac{4aaff}{5b - \cfrac{9aaff}{7b - \cfrac{16aaff}{9b - \text{etc.}}}}}$$

whence, since $\frac{b+af}{b-af} > 1$, it is plain that the value of this expression will be positive.

§14 But if b was a negative number or if one writes $-b$ instead of b, it still must be $b > af$; but then it will be

$$\Delta = \frac{1}{2f}\log\frac{b-af}{b+af},$$

which logarithm will therefore be negative, or

$$\Delta = -\frac{1}{2f}\log\frac{b+af}{b-af},$$

whence one will obtain the following equation

$$\frac{-2af}{\log\frac{b+af}{b-af}} = -b - \cfrac{aaff}{-3b - \cfrac{4aaff}{-5b - \cfrac{9aaff}{-7b - \cfrac{16aaff}{-9b - \text{etc.}}}}}$$

or, having changed the signs,

$$\frac{2af}{\log\frac{b+af}{b-af}} = b + \cfrac{aaff}{-3b + \cfrac{4aaff}{5b + \cfrac{9aaff}{-7b + \cfrac{16aaff}{9b + \text{etc.}}}}}$$

the sum of which continued fraction is therefore equal to that one we found in the preceding paragraph. But that equality of these two expressions will become clear soon to anyone doing the calculation.

§15 In like manner, let us expand the case in which $c = -gg$, for which we found above

$$\Delta = \frac{1}{g}\arcsin\frac{ag}{\sqrt{bb+aagg}},$$

which value expressed in terms of a cosine will give

$$\Delta = \frac{1}{g}\arccos\frac{b}{\sqrt{bb+aagg}},$$

whence it is plain that via the tangent that value will become even simpler; of course,

$$\Delta = \frac{1}{g}\arctan\frac{ag}{b},$$

for which reason for this case this summation results

$$\frac{ag}{\arctan\frac{ag}{b}} = b + \cfrac{aagg}{3b + \cfrac{4aagg}{5b + \cfrac{9aagg}{7b + \cfrac{16aagg}{9b + \text{etc.}}}}}$$

where no restriction is necessary anymore.

§16 Now let us also consider some special cases contained in each of both forms, and since we already observed that the two form in § 13 and 14 are identical, let us use the first, which was

$$\frac{2af}{\log\frac{b+af}{b-af}} = b - \cfrac{aaff}{3b - \cfrac{4aaff}{5b - \cfrac{9aaff}{7b - \text{etc.}}}}$$

and first let us consider the case $b = af$, in which the sum of the continued fraction becomes

$$\frac{2af}{\log\frac{b+af}{b-af}} = 0 = b - \cfrac{bb}{3b - \cfrac{4bb}{5b - \cfrac{9bb}{7b - \text{etc.}}}}$$

which by reduction is easily changed into this one

$$0 = 1 - \cfrac{1}{3 - \cfrac{4}{5 - \cfrac{9}{7 - \cfrac{16}{9 - \text{etc.}}}}}$$

§17 Therefore, for that form to be equal to zero it is necessary that the denominator of the first fraction is $= 1$ and hence

$$1 = 3 - \cfrac{4}{5 - \cfrac{9}{7 - \text{etc.}}} \quad \text{or} \quad 0 = 2 - \cfrac{4}{5 - \cfrac{9}{7 - \text{etc.}}}$$

Therefore, here for the same reason it is necessary that the first denominator becomes $= 2$, such that

$$2 = 5 - \cfrac{9}{7 - \cfrac{16}{9 - \text{etc.}}} \quad \text{or} \quad 0 = 3 - \cfrac{9}{7 - \cfrac{16}{9 - \text{etc.}}}$$

Again here the first denominator must be $= 3$ and hence

$$3 = 7 - \cfrac{16}{9 - \cfrac{25}{11 - \text{etc.}}} \quad \text{or} \quad 0 = 4 - \cfrac{16}{9 - \cfrac{25}{11 - \text{etc.}}}$$

Therefore, again the first denominator must be $= 4$ such that

$$4 = 9 - \cfrac{25}{11 - \text{etc.}}$$

and this way it plain that that relation in like manner holds for infinity, which itself is the criterion for the validity of this equation.

§18 Since in this form the number b must be greater than af, let us now set $b = 2af$ and we will obtain the following summation

$$\frac{2af}{\log 3} = 2af - \cfrac{aaff}{6af - \cfrac{4aaff}{10af - \cfrac{9aaff}{14af - \text{etc.}}}}$$

which is reduced to this mere numerical form

$$\frac{2}{\log 3} = 2 - \cfrac{1}{6 - \cfrac{4}{10 - \cfrac{9}{14 - \cfrac{16}{18 - \text{etc.}}}}}$$

§19 In like manner all letters can be thrown out of the calculation, if one takes a multiple of af for b. For, in general let $b = naf$ and it results

$$\frac{2af}{\log \frac{n+1}{n-1}} = naf - \cfrac{aaff}{3naf - \cfrac{4aaff}{5naf - \cfrac{9aaff}{7naf - \text{etc.}}}}$$

which fraction is reduced to the following form

$$\frac{2}{\log\frac{n+1}{n-1}} = n - \cfrac{1}{3n - \cfrac{4}{5n - \cfrac{9}{7n - \text{etc.}}}}$$

whence it is understood, how all logarithms can be expressed in terms of continued fractions.

§20 Here one could assume fractional numbers for n, but then the first terms in each member would be fractions, which could be reduced to integers; but cases of this kind can most easily be derived from the general form by writing $b = n$ and $af = m$ immediately; for, then we will have

$$\frac{2m}{\log\frac{n+m}{n-m}} = n - \cfrac{mm}{3n - \cfrac{4mm}{5n - \cfrac{9mm}{7n - \text{etc.}}}}$$

whence, if one writes $\sqrt{k}$ instead of m, it will be

$$\frac{2\sqrt{k}}{\log\frac{n+\sqrt{k}}{n-\sqrt{k}}} = n - \cfrac{k}{3n - \cfrac{4k}{5n - \cfrac{9k}{7n - \text{etc.}}}}$$

§21 Therefore, hence we will be able to express the hyperbolic logarithms of all integer numbers as continued fractions. Therefore, in general let i be an integer number and set $\frac{n+m}{n-m} = i$; it will be $\frac{m}{n} = \frac{i+1}{i-1}$. Therefore, take $n = i+1$ and $m = i-1$ and we will have

$$\frac{2(i-1)}{\log i} = i + 1 - \cfrac{(i-1)^2}{3(i+1) - \cfrac{4(i-1)^2}{5(i+1) - \cfrac{9(i-1)^2}{7(i+1) - \cfrac{16(i-1)^2}{9(i+1) - \text{etc.}}}}}$$

whence we conclude

$$\log i = \cfrac{2(i-1)}{i+1-\cfrac{(i-1)^2}{3(i+1)-\cfrac{4(i-1)^2}{5(i+1)-\cfrac{9(i-1)^2}{7(i+1)-\text{etc.}}}}}$$

§22 If we desire the fractions for logarithms of fractional numbers, let us set $\frac{n+m}{n-m} = \frac{p}{q}$, whence $n = p + q$ and $m = p - q$, for which reason we will have

$$\log \frac{p}{q} = \cfrac{2(p-q)}{1(p+q)-\cfrac{1(p-q)^2}{3(p+q)-\cfrac{4(p-q)^2}{5(p+q)-\cfrac{9(p-q)^2}{7(p+q)-\text{etc.}}}}}$$

which form is even more remarkable since it can be conveniently applied to find logarithms approximately. But these fractions will converge the more, the smaller the fraction $\frac{p-q}{p+q}$ was.

§23 To illustrate this in an example, let us take $p = 2$ and $q = 1$, whence certainly only slow convergence is to expected, and it will be

$$\log 2 = \cfrac{2}{3-\cfrac{1}{9-\cfrac{4}{15-\cfrac{9}{21-\text{etc.}}}}}$$

whence by taking only the first term $\frac{2}{3}$ in decimals $0,666666$ results, while from tables one has $\log 2 = 0,693147$, where the error is already sufficiently small. Now let us take the first two terms

$$\cfrac{2}{3-\cfrac{1}{9}} = \frac{9}{13} = 0,6923$$

But taking three terms we will have

$$\frac{2}{3-\cfrac{1}{9-\cfrac{4}{15}}}=\frac{2}{3-\cfrac{15}{131}}=\frac{262}{378}=0,693121,$$

which value deviates from the truth by the quantity $0,000026$.
But a much faster convergence will be detected, if we take $p=3$ and $q=2$, that we have

$$\log\frac{3}{2}=\cfrac{2}{5-\cfrac{1}{15-\cfrac{4}{25-\cfrac{9}{35-\text{etc.}}}}}$$

whose first term gives $\frac{2}{5}=0,400000$; but it actually is $\log\frac{3}{2}=0,405465108$. But having taken two terms

$$\frac{2}{5-\cfrac{1}{15}}$$

one concludes $\log\frac{3}{2}=0,40540$, where the error occurs only in the fifth digit. Take three terms

$$\frac{2}{5-\cfrac{1}{15-\cfrac{4}{25}}}=\frac{2}{5-\cfrac{25}{371}}=0,4054645,$$

where the error occurs just in the seventh digit.

§24 Because of this immense use, which was not expected, it will be worth one's while to make such an investigation in general; and to this end, let us use the formula among the letters m and n given above in § 20, where

$$\log\frac{n+m}{n-m}=\cfrac{2m}{n-\cfrac{mm}{3n-\cfrac{4mm}{5n-\cfrac{9mm}{7n-\cfrac{16mm}{9n-\text{etc.}}}}}}$$

whence, if we take only the first term, it will approximately be

$$\log\frac{n+m}{n-m}=\frac{2m}{n};$$

but having taken the first two terms

$$\cfrac{2m}{n-\cfrac{mm}{3n}}$$

it will more accurately be

$$\log\frac{n+m}{n-m}=\frac{6mn}{3nn-mm},$$

but having taken three terms it will be

$$\log\frac{m+n}{n-m}=\cfrac{2m}{n-\cfrac{mm}{3n-\cfrac{4mm}{5n}}}=\frac{30mnn-8m^3}{15n^3-9mmn}.$$

§25 Indeed, it is not a lot of work to continue these fractions; for, let us put the fraction $\frac{0}{1}$ in front of the ones already found, that we obtain this progression of fractions

$$\begin{array}{cccc} \text{I} & \text{II} & \text{III} & \text{IV} \\ \frac{0}{1}, & \frac{2n}{m}, & \frac{6mn}{3nn-mm}, & \frac{30mnn-8m^3}{15n^3-9mmn}, \end{array}$$

so the numerators as denominators of which can be formed analogously to the case of recurring series. Of course, the third is formed from the first and the second by means of the scale of relation $3n$, $-mm$; the fourth on the other hand is formed from the preceding two by means of the scale of relation $5n$, $-4mm$. For the fifth one has to use the scale of relation $7n$, $-9mm$, for the sixth $9n$, $-16mm$, and so forth. Therefore, this way one easily finds the fifth fraction

$$\text{V} = \frac{210mn^3 - 110m^3n}{105n^4 - 90mmnn + 9m^4},$$

in like manner,

$$\text{VI} = \frac{1890mn^4 - 1470m^3nn + 128m^5}{945n^5 - 1050mmn^3 + 225m^4n}$$

etc.

§26 Here it will be especially helpful to have noted that these fractions increase and get to the truth in continuously smaller increments. But these increments proceed in an extraordinary pattern, as is can easily be seen here:

$$\begin{aligned}
\text{II} - \text{I} &= \frac{2m}{n}, \\
\text{III} - \text{II} &= \frac{2m^3}{n(3nn - mm)}, \\
\text{IV} - \text{III} &= \frac{2 \cdot 4m^5}{(3nn - mm)(15n^3 - 9mmn)}, \\
\text{V} - \text{IV} &= \frac{2 \cdot 4 \cdot 9m^7}{(15n^3 - 9mmn)(105n^4 - 90mmnn + 9m^4)}, \\
\text{VI} - \text{V} &= \frac{2 \cdot 4 \cdot 9 \cdot 16m^9}{(105n^4 - 90mmnn + 9m^4)(945n^5 - 1050mmn^3 + 225m^4n)},
\end{aligned}$$

whence it is plain that the larger the number n was compared to m the faster these differences become so small that they can be neglected without error.

§27 From § 15 the circular arc whose tangent is $\frac{ag}{b}$ is expressed by a continued fraction in such a way that

$$\arctan\frac{ag}{b} = \cfrac{ag}{b + \cfrac{aagg}{3b + \cfrac{4aagg}{5b + \cfrac{9aagg}{7b + \text{etc.}}}}}$$

Now, analogously to the above formulas, let us put $ag = m$ and $b = n$ and so we will have

$$\arctan\frac{m}{n} = \cfrac{m}{n + \cfrac{mm}{3n + \cfrac{4mm}{5n + \cfrac{9mm}{7n + \text{etc.}}}}}$$

which form converges the faster the greater the number n was compared to m; hence it is plain that this expression can fruitfully be accommodated to calculations.

§28 Let us start from the case in which $m = 1$ and $n = 1$ and in which

$$\arctan\frac{m}{n} = \frac{\pi}{4} = \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{4}{5 + \cfrac{9}{7 + \cfrac{16}{9 + \text{etc.}}}}}}$$

which fraction certainly does not converge that fast; but nevertheless, let us see how close it comes to the truth, since we know that $\frac{\pi}{4} = 0,7853816339$. And the first term will give

$$\frac{\pi}{4} = \frac{1}{1} \quad \text{(too large)};$$

two terms yield

$$\frac{\pi}{4} = \cfrac{1}{1+\cfrac{1}{3}} = \frac{3}{4} \quad \text{(too small);}$$

three terms give

$$\frac{\pi}{4} = \cfrac{1}{1+\cfrac{1}{3+\cfrac{4}{5}}} = \frac{19}{24} = 0,7916 \quad \text{(too large).}$$

Take four terms that

$$\frac{\pi}{4} = \cfrac{1}{1+\cfrac{1}{3+\cfrac{4}{5+\cfrac{9}{7}}}} = \frac{40}{51} = 0,7853 \quad \text{(too small),}$$

where the error is just detected in the third figure.
Furthermore, this continued fraction is similar to that one Brouncker once gave, which reads as follows

$$\cfrac{1}{2+\cfrac{9}{2+\cfrac{25}{2+\cfrac{49}{2+\text{etc.}}}}}$$

It is manifest that our fraction converges a lot faster; and it is not less beautiful.

§29 But to obtain a faster-converging continued fraction, let us set $\arctan \frac{m}{n} = 30^\circ$, since the tangent of which is $\frac{1}{\sqrt{3}}$, for the number n not to become irrational, let us take $m = \sqrt{3}$ and $n = 3$; therefore, hence it will be

$$\frac{\pi}{6} = \cfrac{\sqrt{3}}{3 + \cfrac{3}{9 + \cfrac{12}{15 + \cfrac{27}{21 + \cfrac{48}{27 + \text{etc.}}}}}}$$

which form is reduced to the following

$$\frac{\pi}{6\sqrt{3}} = \cfrac{1}{3 + \cfrac{1}{3 + \cfrac{4}{15 + \cfrac{9}{7 + \cfrac{16}{27 + \cfrac{25}{11 + \text{etc.}}}}}}}$$

for the expansion of which we want to find the value $\frac{\pi}{6\sqrt{3}}$ approximately, which is $0,3022998$. But now the first term yields

$$\frac{\pi}{6\sqrt{3}} = 0,3333;$$

but the first two yield

$$\frac{\pi}{6\sqrt{3}} = \cfrac{1}{3 + \cfrac{1}{3}} = \frac{3}{10} = 0,3000;$$

three terms give

$$\frac{\pi}{6\sqrt{3}} = \cfrac{1}{3 + \cfrac{1}{3 + \cfrac{4}{15}}} = \frac{49}{162} = 0,30247,$$

where the error affects just the fourth digit.

§30 But a faster convergence can be obtained by splitting the right angle into two parts, as I once showed to be

$$\arctan\frac{1}{2}+\arctan\frac{1}{3}=\arctan 1=\frac{\pi}{4}.$$

Therefore, so we will find two continued fractions, whose sum will give the value of $\frac{\pi}{4}$, which will be

$$\arctan\frac{1}{2}=\cfrac{1}{2+\cfrac{1}{6+\cfrac{4}{10+\cfrac{9}{14+\text{etc.}}}}}\quad\text{and}\quad\arctan\frac{1}{3}=\cfrac{1}{3+\cfrac{1}{9+\cfrac{4}{15+\cfrac{9}{21+\text{etc.}}}}}$$

But it is manifest that both these fractions, and especially the second, converge rapidly.

§31 But let us convert even our general continued fraction into common ones; and from the first term only we find

$$\arctan\frac{m}{n}=\frac{m}{n};$$

from two terms it results

$$\arctan\frac{m}{n}=\frac{3mn}{3nn+mm};$$

three terms yield

$$\arctan\frac{m}{n}=\frac{15mnn+4m^3}{15n^3+9mmn}.$$

Take four terms, whence

$$\arctan\frac{m}{n}=\frac{105mn^3+55m^3n}{105n^4+90mmnn+9m^4}.$$

If now as above the fraction $\frac{0}{1}$ is put in front of these fractions, this progression will arise

$$\begin{array}{ccccc} \text{I} & \text{II} & \text{III} & \text{IV} & \text{V} \\ \dfrac{0}{1}, & \dfrac{m}{n}, & \dfrac{3mn}{3nn+mm}, & \dfrac{15mnn+4m^3}{15n^3+9mmn}, & \dfrac{105mn^3+55m^3n}{105n^4+90mmnn+9m^4}, \end{array}$$

each term of which can likewise be formed from the preceding two according to a certain law, of course,

for	III	the scale of relation is	$3n$,	$+mm$
for	IV	the scale of relation is	$5n$,	$+4mm$
for	V	the scale of relation is	$7n$,	$+9mm$
		etc.		